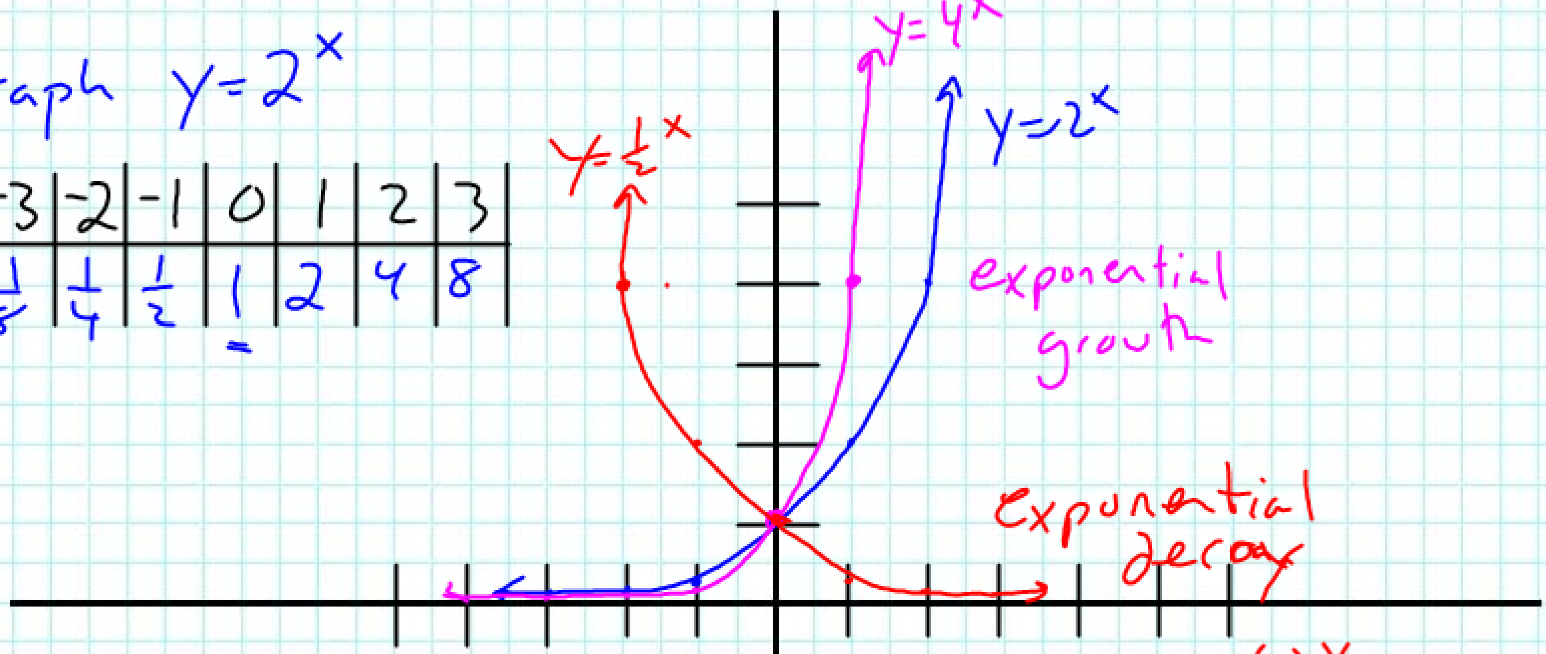
Exponential Function

$y = ab^x$, $a \neq 0$, $b > 0$, $b \neq 1$

Graph $y = 2^x$

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



Graph $y = 4^x$

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64

Graph $y = \left(\frac{1}{2}\right)^x$

x	-3	-2	-1	0	1	2	3
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

If $f(x) = 3^x$, sketch NO CALC.

(a) $g(x) = 3^{x+1}$ $f(x+1)$ horizontal shift
lft 1

(b) $g(x) = 3^x - 2$ $f(x) - 2$ vertical shift
down 2

(c) $g(x) = -3^x$ $-f(x)$ reflect over x-axis

(d) $g(x) = 3^{-x} \Rightarrow \frac{1}{3^x}$ $f(-x)$ reflect over y-axis

Growth factor $b = 1 + r$, r = rate of increase

$y = ab^x$

start value

often time

Exponential Decay $b < 1$

$b = (1 - \text{rate of decrease})$

6% decrease

$1000(1 - 0.06)^x$

$1000(0.94)^x$

$\cancel{1000(0.06)^x}$

Investments

- APR → Annual percentage rate   %

- compounding

  monthly $\frac{APR}{12}$, your x is now months

Invest \$5000 at 4 % APR compounded monthly

$$y = a\,b^{x}$$

a → start, b → $1+r$

$$y = 5000\left(1+\frac{0.04}{12}\right)^{x} \quad x \to \text{months}$$

or

$$y = 5000\left(1+\frac{0.04}{12}\right)^{12x} \quad x = \text{yrs}$$

You invest \$1.00 at 100% APR for 1 year. What is your balance at the end of the year if you compound the interest

Keep lots of decimal places

(a) Yearly? 2.00

(b) Quarterly? 2.4414

(c) Monthly? 2.61303

(d) Weekly? 2.6926

(e) Daily? 2.7146

(f) Every hour? 2.7181

(g) Every minute? 2.718279 $\longrightarrow \approx e$

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

e - natural base

$$A = P(1+r)^t$$

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$m = \frac{n}{r} \qquad \frac{1}{m} = \frac{r}{n}$$

$$n = mr$$

$$A = P\left(1 + \frac{1}{m}\right)^{mrt}$$

$$A = P\left(\underline{\left(1 + \frac{1}{m}\right)^m}\right)^{rt}$$

$$A = Pe^{rt}$$

r → rate (like 0.05)

t → time in yrs.

A → balance

P → initial investment

Continuous Compounding

To compound continuously, we use the natural base (e). We most often use this with natural events of growth (e.g. population) or decay.

$$y = ae^{bx}$$

or as we did in class for an investment

$$A = Pe^{rt}$$

A → ending amount; P → start value (principal); e → base e; r → the rate; t → usually years

Ex. \$3,000 invested at 4.5% APR compounded continuously.

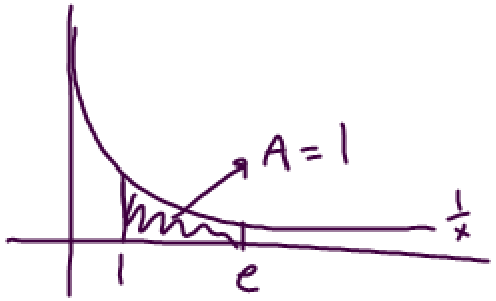
$$A = 3000e^{(0.045 \cdot t)}$$

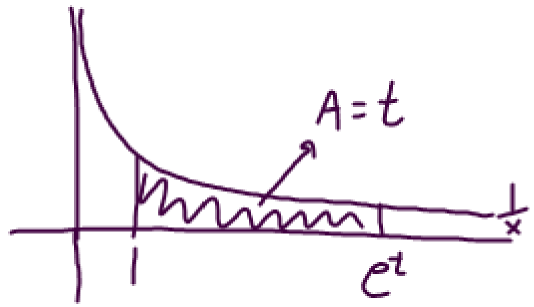
For 10 years later: $A = 3000e^{(0.045 \cdot 10)}$

$$\boxed{\approx \$4,704.94}$$

More on e and e^x

- Area bounded by $f(x)=\frac{1}{x}$ and x-axis and 1 and e $= 1$
  and 1 and $e^t = t$





- derivitive and integral of $e^x = e^x$

- you can change any equation $y=ab^x$ to base e by writing b in the form $b=e^{\ln b}$

For example, $y=3(2)^x \Rightarrow y=3e^{(\ln 2)x} \approx 3e^{0.693x}$

Half-Life

You have $75,000 in a retirement account.

Your account loses half its value every 5 years.

(a) Write a model for this situation.

$y = ab^x$

$75000(1-0.5)^{\frac{x}{5}}$ – every 5 yr's

(b) Find the value after 9 years.

21,538

The half-life of a radioactive substance is the time it takes for half of the material to decay. A hospital prepares a 100mg supply of technetium-99m, which has a half-life of 6 hours. Write an exponential function for the amount of technetium-99m after x hours and then find the amount remaining after 75 hours.

$$f(x) = 100\left(\frac{1}{2}\right)^{\frac{x}{6}} \quad x \rightarrow \text{hours}$$

$$f(75) = 100\left(\frac{1}{2}\right)^{\frac{75}{6}} \approx 0.02\text{mg}$$

Writing an exponential equation through two points

$y = ab^x \quad (2, 2)\ (3, 4)$

1. $2 = ab^2$

2. $\frac{2}{b^2} = \frac{ab^2}{b^2}$

$\frac{2}{b^2} = a$

3. $y = ab^x$

$4 = \frac{2}{b^2} \cdot b^3$

$4 = \frac{2b^3}{b^2}$

$\frac{4}{2} = \frac{2b}{2}$

$2 = b$

4. $\frac{2}{b^2} = a \rightarrow \frac{2}{2^2} = a$

$a = \frac{1}{2}$

Steps

1. Start with $y = ab^x$, plug in first pt. for x + y
2. Solve for a
3. Plug in a, and 2nd pt. $y = ab^x$ solve for b
4. Sub. your b back into the a equation to find a
5. Write equation using a + b

$$\boxed{y = \frac{1}{2}(2)^x}$$

Write an equation, $y = ab^x$, through $(2,4)(3,16)$

Logarithmic Function

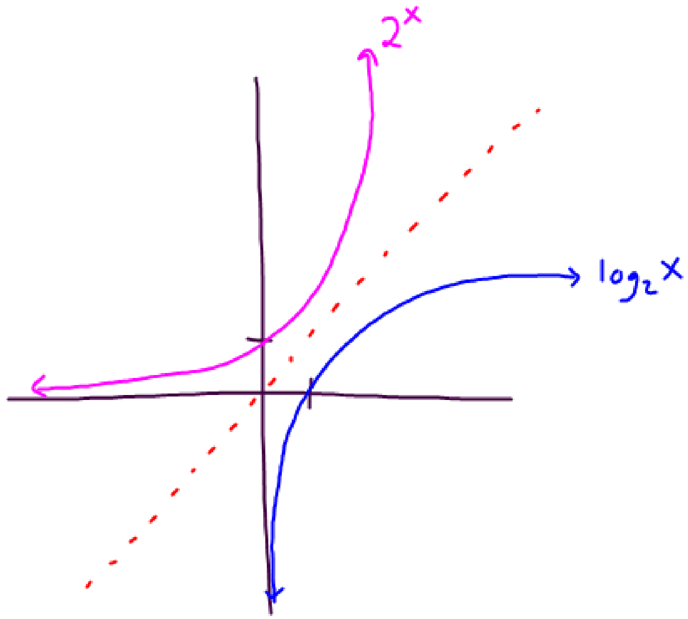
$y = \log_a x$ if and only if $x = a^y$

$x > 0,\ a > 0,\ a \neq 1$

$f(x) = \log_a x \rightarrow$ value (Ans to exponential equation)

$f(x)$ ↓ exponent

a ↓ base



* The log is the inverse of the exponential function

* If it just has $y = \log x$ with no base listed, the base is 10

* $y = \log_e x \Rightarrow y = \ln x$

Properties of Logs - Basic

① $\log_a 1 = 0$ b/c $a^0 = 1$

② $\log_a a = 1$ b/c $a^1 = a$

③ $\log_a a^x = x$ b/c $a^{\log_a x} = x$

④ If $\log_a x = \log_a y$ then $x = y$

All these apply to natural log, ln, as well

More Properties

① $\log_a(uv) = \log_a(u) + \log_a(v)$

② $\log_a\left(\frac{u}{v}\right) = \log_a(u) - \log_a(v)$

③ $\log_a(u^n) = n\log_a(u)$

All the same properties hold with the natural log, ln

Examples

① Expand

$\log_4 5x^3y$

$\downarrow$

$\log_4 5 + \log_4 x^3 + \log_4 y$

$= \log_4 5 + 3\log_4 x + \log_4 y$

② Condense

$2\ln(x+2) - \ln x$

$\ln(x+2)^2 - \ln x$

$= \ln \dfrac{(x+2)^2}{x}$

You Try

① Expand $\ln \dfrac{\sqrt{3x-5}}{7}$

② Condense $\frac{1}{3}\left[\log_2 x + \log_2(x-4)\right]$

All you really need

$7^x = 12$ exponential form

$\log_7(12) = x$ logarithmic form

$$x = \frac{\log 12}{\log 7}$$ Ans

To graph $\log_5 x$ on calc $\rightarrow$ $y = \dfrac{\log x}{\log 5}$

① In a research experment, a population of fruit flies is increasing according to the law of exponential growth. After 2 days there are 100 flies, and after 4 days there are 300 flies. Find a model for this situation in the form

$y = ab^x$ and $y = ae^{bx}$. Get two pts. $(2, 100)(4, 300)$

① Solve for a using 1st pt

$y = ab^x$

$100 = ab^2$

$a = \frac{100}{b^2}$

② Solve for b using 2nd pt

$y = ab^x$

$300 = \frac{100}{b^2} \cdot b^4$

$300 = \frac{100b^4}{b^2}$

$300 = 100b^2$

$3 = b^2$

$b = \sqrt{3}$

③ Use b to find a

$a = \frac{100}{b^2}$

$a = \frac{100}{(\sqrt{3})^2}$

$a = \frac{100}{3}$

③ Write equation

$\boxed{y = \frac{100}{3}(\sqrt{3})^x}$

For $y = ae^{kx}$ either solve as above or use $k = \ln b$

$\boxed{y = \frac{100}{3}e^{0.5493x}}$

② On a college campus of 5,000 students, one student returns from vacation with a virus. The spread of the virus is modeled by

$$y = \frac{5000}{1 + 4999e^{-0.8(x)}}$$

x = # of days
y = # ♀ infected

ⓐ How many people are infected after 5 days?

ⓑ How many days until 40% of the students are infected?

ⓐ $f(5) = \frac{5000}{1+4999e^{-0.8(5)}}$

$\boxed{f(5) \approx 54 ♀}$

ⓑ 40% of 5000 = 2000 so,

$(1+4999\ldots)\ 2000 = \frac{5000}{1+4999e^{-0.8x}} \cdot (1+4999e^{-0.8x})$

$2000(1+4999e^{-0.8x}) = 5000 \Rightarrow 1+4999e^{-0.8x} = \frac{5}{2}$

$4999e^{-0.8x} = 1.5 \Rightarrow e^{-0.8x} = \frac{1.5}{4999}$

$-0.8x = \ln\left(\frac{1.5}{4999}\right) \Rightarrow x = \frac{\ln\left(\frac{1.5}{4999}\right)}{-0.8} \approx \boxed{10 \text{ days}}$

Solve for x

① $e^x = 27$

$$\boxed{\begin{aligned} x &= \ln 27 \\ &\approx 3.296 \end{aligned}}$$

② $3e^{2x} - 7 = 1$

$+7 \qquad +7$

$\frac{3e^{2x}}{3} = \frac{8}{3}$

$e^{2x} = \frac{8}{3}$

$2x = \ln \frac{8}{3}$

$$\boxed{\begin{aligned} x &= \frac{\ln \frac{8}{3}}{2} \\ x &\approx 0.4904 \end{aligned}}$$

③ $\ln x = 3$

$\ln_e x = 3$

$$\boxed{\begin{aligned} e^3 &= x \\ x &\approx 20.086 \end{aligned}}$$

④ $\log_5(3x-2) = \log_5(x+4)$

one-to-one property

$3x - 2 = x + 4$

$+2 \qquad +2$

$3x = x + 6$

$-x \qquad -x$

$2x = 6$

$\boxed{x = 3}$

⑤ $e^{2x} - 9e^x + 14 = 0$

$(e^x)^2 - 9e^x + 14 = 0$

like $x^2 - 9x + 14 = 0$

$(e^x - 7)(e^x - 2) = 0$

$e^x = 7 \qquad e^x = 2$

$$\boxed{\begin{aligned} x &= \ln 7 & x &= \ln 2 \\ &\approx 1.9459 & &0.6932 \end{aligned}}$$

⑥ $\ln(x-3) + \ln(x+3) = 1$

product property

$\ln\left((x-3)(x+3)\right) = 1$

$\ln(x^2 - 9) = 1$

$e^1 = x^2 - 9$

$+9 \qquad +9$

$e + 9 = x^2$

$x = \pm\sqrt{e+9}$

$\approx \pm 3.423$

But -3.423 is extraneous

$x \approx 3.423$ only answer